Frank-Wolfe Methods for Minimizing Log-Homogeneous Self-Concordant Barriers

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1. Introduction
2. Applications
3. Our Method: New Generalized Frank-Wolfe Method
4. Computational Experiments

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- Includes many applications (coming up later).

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  2. $f(u_k) \to \infty$ for any $\{u_k\}_{k\geq 1} \subseteq \mathsf{int}\,\mathcal{K}$ such that $u_k \to u \in \mathsf{bd}\,\mathcal{K}$,
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  - $f(u) = -\sum_{j=1}^m w_j \ln(u_j)$ for $u \in \mathcal{K} := \mathbb{R}_+^m$ and $\theta = \sum_{j=1}^m w_j$ where $w_1, \ldots, w_n \geq 1$.

A Motivating Example: D-optimal Design

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- Arises in many places, including optimal experimental design, and as the dual problem of the minimum volume enclosing ellipsoid problem.
- Khachiyan (1996) proposed a "barycentric coordinate ascent" method with exact line-search, which is actually FW with exact line-search. Method works remarkably well both in theory and practice: it computes an ε-optimal solution of (D-OPT) in (essentially) $O(n^2/\varepsilon)$ iterations.

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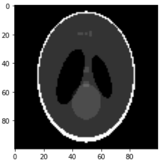
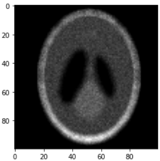
- What problem structure actually drives the success of Khachiyan's method? And might such structure exist anywhere else?
- We resolve this mystery and generalize his method to the much broader class of problems in (P), even while relaxing the exact line-search requirement.

Poisson Image Deblurring with TV Regularization



True image X



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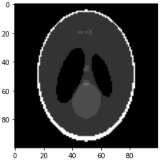

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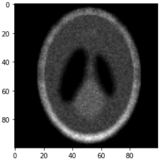

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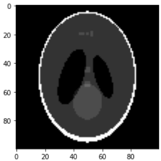

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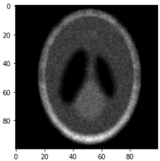

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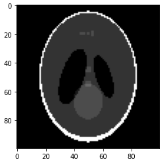

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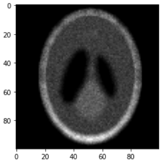

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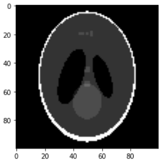

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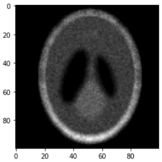

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- For convenience, we also represent A in its matrix form $A \in \mathbb{R}^{N \times N}$, where $N := mn$, and vectorize Y and X into $y \in \mathbb{R}^N$ and $x \in \mathbb{R}^N$, respectively. Notation: we write $x = \mathsf{vec}(X)$ and $X = \mathsf{mat}(x)$, etc.

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- We seek to recover X from Y (equivalently x from y) using maximum-likelihood estimation on the TV-regularized problem:

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- (`Deblur`) has a (standard) total-variation (TV) regularization term to recover a smooth image with sharp edges. The TV term is given by

$$\begin{aligned} \mathrm{TV}(x) := & \textstyle\sum_{i=1}^m \sum_{j=1}^{n-1} \left|[\mathsf{mat}(x)]_{i,j} - [\mathsf{mat}(x)]_{i,j+1}\right| \\ & + \textstyle\sum_{i=1}^{m-1} \sum_{j=1}^{n} \left|[\mathsf{mat}(x)]_{i,j} - [\mathsf{mat}(x)]_{i+1,j}\right|. \end{aligned}$$

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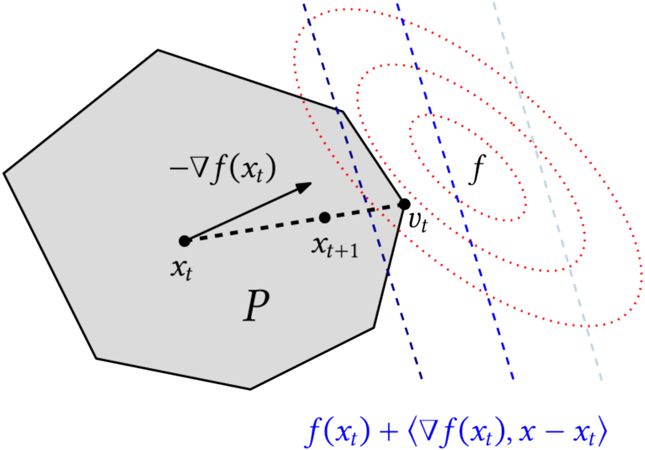
- Computational geometry (computing the analytic center of a polytope)

A Simple Illustration When $h = \iota_{\mathcal{P}}$



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- **Repeat** (until some convergence criterion is met)

$$v^k \in \arg\min_{x\in\mathbb{R}^n} \langle \nabla f(\mathsf{A}x^k), \mathsf{A}x\rangle + h(x) \qquad \text{(Solve Lin. subproblem)}$$

$$G_k := \langle \nabla f(\mathsf{A}x^k), \mathsf{A}(x^k - v^k)\rangle + h(x^k) - h(v^k) \qquad \text{(FW-Gap)}$$

$$D_k := \|\mathsf{A}(v^k - x^k)\|_{\mathsf{A}x^k} \qquad \text{(Local Distance)}$$

$$\alpha_k := \min\left\{\frac{G_k}{D_k(G_k + D_k)}, 1\right\} \qquad \text{(Stepsize)}$$

$$x^{k+1} := x^k + \alpha_k(v^k - x^k) \qquad \text{(Update)}$$

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- ▷ For some applications (e.g., PET and D-optimal design), the step-size can also be efficiently computed via exact line-search.

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- (Iteration complexity for ε-optimality gap) Let K_ε denote the number of iterations required by gFW-LHSCB to obtain $\delta_k \le \varepsilon$. Then:

$$K_\varepsilon \le \lceil 5.3(\delta_0 + \theta + R_h) \ln(10.6\delta_0) \rceil + \left\lceil 12(\theta + R_h)^2 \max\left\{ \frac{1}{\varepsilon} - \frac{1}{\delta_0} \; , 0 \right\} \right\rceil \; .$$

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- (Iteration complexity for ε-FW gap) Let $\mathrm{FWGAP}_\varepsilon$ denote the number of iterations required by gFW-LHSCB to obtain $G_k \le \varepsilon$. Then:

$$\mathrm{FWGAP}_\varepsilon \le \lceil 5.3(\delta_0 + \theta + R_h) \ln(10.6\delta_0) \rceil + \left\lceil \frac{24(\theta + R_h)^2}{\varepsilon} \right\rceil \, .$$

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For many applications, all of the three quantities can be easily estimated, and hence the computational guarantees are known before running the algorithm.

Computational Experiments on Poisson Image Deblurring with TV Regularization

$$\min_{x\in\mathbb{R}^N} \;\; \bar{F}(x) := \underbrace{-\textstyle\sum_{l=1}^N y_l \ln(a_l^\top x)}_{=f(\mathsf{A}x)} + \underbrace{\langle \textstyle\sum_{l=1}^N a_l, x\rangle + \lambda \mathrm{TV}(x)}_{=h(x)} \qquad \texttt{(Deblur)}$$

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▷ Since $\mathrm{TV}(\cdot)$ is polyhedral, and the linear-optimization sub-problem

$$v^k \in \arg\min_{0\le x\le Me} \langle \nabla f(\mathsf{A}x^k), \mathsf{A}x\rangle + \langle \sum_{l=1}^N a_l, x\rangle + \lambda \mathrm{TV}(x)$$

can be formulated as a relatively simple LP and solved easily using a standard LP solver such as Gurobi.

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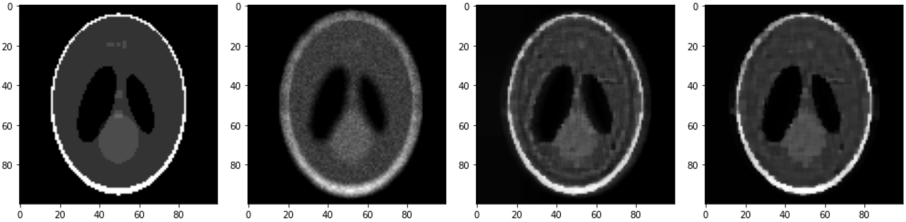
- We chose the starting point $x^0 = \mathsf{vec}(Y)$ (the vectorized noisy image), and we set $\lambda = 0.01$.
- We used CVXPY to (approximately) compute the optimal objective value $\bar{F}^*$ of (`Deblur`) in order to compute optimality gaps.

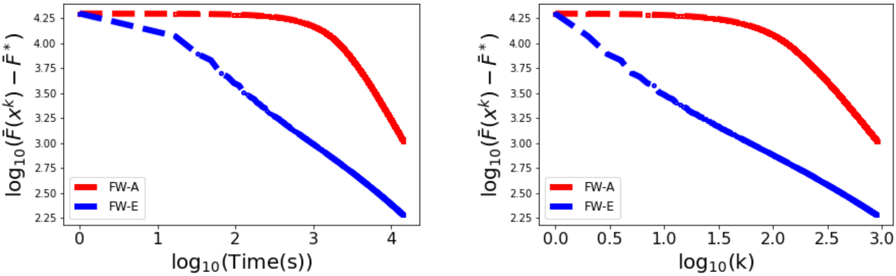
Results: Recovered Images



(k) True image X (l) Noisy image Y (m) `FW-Adapt` (n) `FW-Exact`

Figure 1: True, noisy and recovered Shepp-Logan phantom images.

Results: Optimality Gaps versus Time and Iterations



(a) Optimality gap versus time (in seconds)

(b) Optimality gap versus iterations

Figure 2: Comparison of optimality gaps of `FW-Adapt` (`FW-A`) and `FW-Exact` (`FW-E`) for image recovery of the Shepp-Logan phantom image.

Motivating Example: D-Optimal Design

$$\begin{aligned} \min \;\; & f(x) := -\ln\det\left(\textstyle\sum_{i=1}^m x_i a_i a_i^\top\right) \\ \text{s.t.} \;\; & x \in \Delta_m := \{\textstyle\sum_{i=1}^m x_i = 1,\ x_i \ge 0,\ \forall\, i \in [m]\}. \end{aligned} \tag{D-OPT}$$

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$$i_k \in \arg\min_{i\in[m]} \nabla_i f(x^k), \qquad G_k := -\nabla_{i_k} f(x^k) - n,$$

$$j_k \in \arg\max_{j: x_j^k > 0} \nabla_i f(x^k), \quad \tilde{G}_k := \nabla_{j_k} f(x^k) + n,$$

$$d^k = \begin{cases} e_{i_k} - x^k & \text{if } G_k > \tilde{G}_k \\ x^k - e_{j_k} & \text{otherwise} \end{cases}, \qquad x^{k+1} := x^k + \alpha_k d^k,$$

where the stepsize $\alpha_k \ge 0$ is given by exact line-search.

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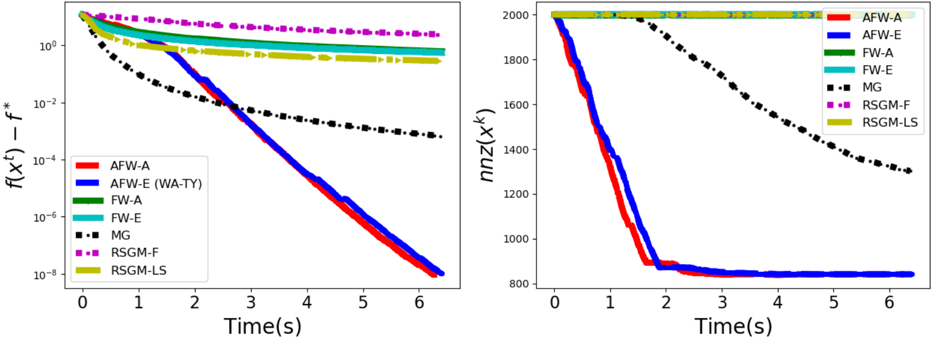
- ▷ Structurally, this method coincides with the Frank-Wolfe method with Wolfe's away-step (1970), and it was rediscovered by Todd and Yıldırım (2005) — therefore, it is referred to as the WA-TY method.
- ▷ Excellent numerical performance:



ASFW-A & ASFW-E (this work): Away-step FW methods for LHB
FW-A & FW-E [Fed72; Kha96; ZFce]: Generalized FW methods for LHB
RSGM-F & RSGM-LS [BBT17; LFN18]: Relatively smooth gradient method
MG [STT78]: Multiplicative gradient method

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- Some deeper questions:
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- In this work, we will provide affirmative answers to the questions above.

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- Besides D-optimal design, other applications include
  - Budget-constrained D-optimal design
  - Positron emission tomography
  - (Reformulated) Poisson image deblurring with TV-regularization

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  - (Away direction) If $|\mathcal{S}_k| > 1$, compute $a^k \in \arg\max_{x \in \mathcal{S}_k} \langle \nabla F(x^k), x \rangle$, $d_{\mathrm{A}}^k := x^k - a^k$ and $\tilde{G}_k := \langle -\nabla F(x^k), d_{\mathrm{A}}^k \rangle$.
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Away-step Frank-Wolfe Method for solving (P)

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- ▷ For all applications of interest, computing $D_k = \|\mathsf{A}d^k\|_{y^k} = \langle \nabla^2 F(x^k)d^k, d^k\rangle^{1/2}$ takes $O(n)$ times, instead of $O(n^2)$ time.

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  where

  - μ is the quadratic-growth constant of f on $\mathcal{Y}$ that only depends on $R_{\mathcal{Y}}(y^*)$
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- ▷ All the quantities defining ρ are *affine-invariant* and *norm-independent*.

Computational Guarantees

Global linear convergence of $\{G_k\}_{k\geq 0}$:

For some (affine-invariant) $\bar{D} < +\infty$ and all $k \geq 0$, we have

$$G_k \leq \begin{cases} 4(1-\rho)^{k_{\text{eff}}}\delta_0 \max\{\bar{D}, 1\}, & \text{if } \delta_k > 1 \\ 4\sqrt{1-\rho}^{k_{\text{eff}}}\sqrt{\delta_0}\max\{\bar{D}, 1\}, & \text{if } \delta_k \leq 1 \end{cases}.$$

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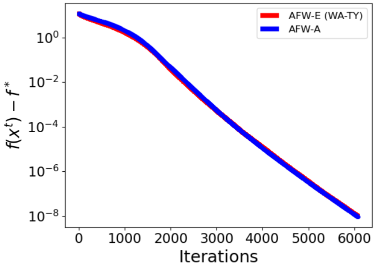

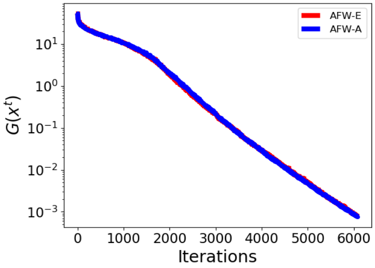

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<u>Land on $\mathcal{F}$ in finite iterations:</u>

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For all $k \geq \bar{k}$, if $x^k \notin \mathcal{F}$, then

- $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$, when either exact line-search or adaptive stepsize is used in Step 26,
- $\mathcal{S}_{k+1} = \mathcal{S}_k \setminus \{a^k\}$ for some $a^k \in \mathcal{S}_k \cap \bar{\mathcal{V}}_{\mathcal{F}}$, when exact line-search is used in Step 26;

otherwise, if $x^k \in \mathcal{F}$, then $x^l \in \mathcal{F}$ for all $l \geq k$.

Another Example: Positron Emission Tomography

$$\max_{x\in\Delta_n}\ \left\{F(x) := \sum_{j=1}^m p_j \ln(a_j^\top x)\right\} \tag{PET}$$

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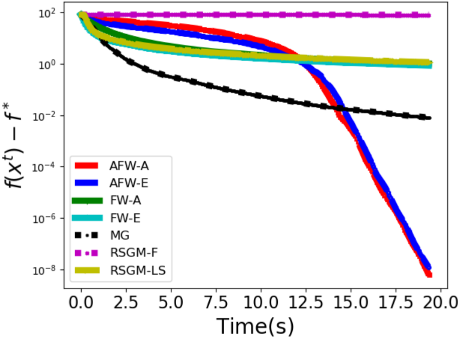

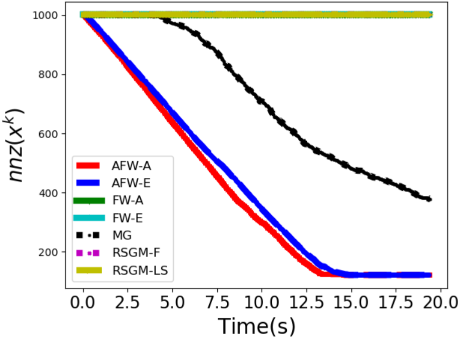

Thank you!

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